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Hysteresis in random field Ising model at zero temperature

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Abstract We discuss hysteresis in the ferromagnetic random field Ising model at zero temperature. It provides a paradigm of the nonequilibrium response of a complex system which possesses infinitely many metastable states but insufficient thermal energy to move between these states over practical time scales

Keywords Ising model, magnetic hysteresis, random processes

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1. Introduction

Statistical description of a dynamical system is generally based on an implicit time scale. Consider for example, a cup of coffee which the author forgot to drink while writing this article. One can use statistical mechanics to describe this system. However, the appropriate model of the system is determined by the time scale of interest. If the coffee has been sitting for a few hours only, it is reasonable to suppose that its mass has not changed much and it is a liquid in thermal equilibrium with its surroundings. Of course, some hot molecules must have left the liquid as it cooled but on the scale of a few hours the dominant effect is the loss of energy, not the loss of matter. After a few weeks, the liquid would evaporate into air completely, and the exchange of mass from inside the cup to outside would become the dominant effect. We would then use the statistical mechanics of a gas to describe the system. This particular system is easy to analyze. But it brings out a general principle which is often tedious to implement. In order to describe the system at time scale τ, it is convenient to use a metastable state of the system in which all relaxation processes which take longer than τ are considered frozen.

The 'frozen' degrees of freedom may occur in a variety of complex configurations, and therefore, the determination of the stationary state in a frozen background is a technically challenging task. For example, the statistical mechanics of spin glasses and ordinary window glasses is quite nontrivial. Exact solutions of models in this class are scarce. In case of our cup of coffee, the liquid formed a metastable state which yielded to more stable gaseous state at longer times. In case of spin glasses, glasses, and an example which we shall discuss in more detail below, there is not a single metastable state but an infinity of metastable states of the system The energy barriers between the metastable states are much higher than available thermal energies. Thus, the system remains trapped in one of its metastable states for the entire period of observation. The observed properties of the system are ensemble averages over the domain of the metastable state *i.e.* a small set of states which may be accessible from each other with available thermal energy. In order to cross large barriers between metastable states, we have to drive the system by an external force which is sufficiently strong to overcome the barriers. In driven systems of this type, history dependent effects are to be expected because the system explores only a limited portion of the phase space in each metastable state.

We now come to a specific system which will take up the remainder of this article. Consider a typical magnet; it could be a ferromagnet or an anti-ferromagnet. We focus on a ferromagnet in the present article. The natural state of a ferromagnet has typically zero magnetization. This is because dipolar, and anisotropy forces compete with exchange forces at large length scales to form domains of closure. *Ab initio* calculations of the domain structure are formidable and it is outside the scope of this article to attempt them. We are content to note that a very large number of domain structures are apparently possible, and

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we take these to be the metastable states of the system. The energy barriers between different metastable states are much larger than thermal energies at room temperature because there is little evidence that domain structures change spontaneously at room temperature over practical time scales. If the magnet is placed in an increasing applied field, favorably oriented domains grow in size at the expense of unfavourably oriented domains. An interesting aspect of this response is that the variation in the magnetization is rough at a microscopic level even if the driving field changes smoothly. The magnetization changes by irregular microscopic jumps known as Barkhausen noise. The Barkhausen noise has some apparently scale-invariant features which we shall study here. It is also a paradigm for the punctuated character of a wide class of nonequilibrium phenomena including earthquakes

Zero-temperature dynamics of the random field Ising model (RFIM) provides an interesting caricature of the nonequilibrium phenomena mentioned above [1,2]. In Section 2, we describe this model. In Section 3, we describe a mean field theory of the model based on infinite-range ferromagnetic interactions, and contrast the predictions of this theory from numerical simulations of the model with short-range ferromagnetic interactions on a cubic lattice. In Section 4, we present another mean field theory of the model in the form of an exact solution of the short-range interaction RFIM on a Bethe lattice of coordination number z Section 4 emphasizes the key ideas behind the exact solution, rather than the details which can be found in the literature [3-7]. It is hoped that Section 4 will pass on to the reader the benefit of a clearer understanding of the mathematical steps which the author has acquired with the passage of time. The main finding of this section is that the model on a Bethe lattice with coordination number z has qualitatively the same behavior as on a regular lattice with the same co-ordination number. The Bethe lattice with $z = 2$ is of course, identical with the one-dimensional lattice [3]. We find that the hysteresis in the RFIM on a Bethe lattice with $z = 3$, 4, and 6 is qualitatively similar to that on a hexagonal, square, and cubic lattice respectively. This is mentioned in the form of concluding remarks in Section 5. We also mention in Section 5, the status of the anti-ferromagnetic RFIM.

2. The model

The RFIM in a uniform external field h is characterized by the Hamiltonian

$$H = -J\sum s_i s_j - \sum h_i s_i - h\sum s_i . \tag{1}$$

The sum in the first term is restricted to pairs of nearest neighbors on a lattice; $s_i = \pm 1$ is an Ising spin at site i, and h_i is a random field drawn from a continuous distribution $p(h_i)$. The external field h is cycled from $-\infty$ to $+\infty$ and back to $-\infty$. This takes the system around its major hysteresis loop. Spins turn up on the lower half of the loop, and turn down again on the upper half We assume that the system is at zero temperature. Thus, the orientation of the spin a[t] site i is determined entirely by the effective local field h_i^{eff} a[t] site i.

$$h_i^{eff} = J\sum s_j + h_i + h . \tag{2}$$

If $h_i^{eff} \geq 0$, we set $s_i = 1$. If $h_i^{eff} < 0$, we set $s_i = -1$. We ca[ll] this the zero-temperature single-spin-flip dynamics. It choose[s] one of the two states of an Ising spin that corresponds to low[er] energy. For a given h, we apply this dynamics iteratively keepin[g] h fixed, till all spins are aligned along the net fields at their site[s]. This corresponds to a local minimum energy state of the systen[m]. It is a stable state when the system is at zero-temperature W[e] assume that it corresponds to a metastable state of the syste[m] at a non-zero temperature. Starting from a stable state at zer[o] temperature, the next stable state is determined by increasing [or] decreasing h as appropriate till some site has to be flippe[d]. After this site is flipped, one or more neighboring sites ma[y] have to be flipped as well. We hold the external field fixed at i[ts] new value, and relax the system till each spin is aligned along [its] net field again. The number of spins that have to be flipped i[n] this process determines the size of the avalanche. Holding th[e] external field fixed during an avalanche amounts to an assumptio[n] that the external field changes infintely slowly in compariso[n] with the rate at which individual spins relax In numeric[al] simulations, the values of h where instabilities occur, and th[e] number of spins which have to be flipped to repair ea[ch] instability (the size of avalanche) are random events dependin[g] upon the size of the system, and the realized configuration [of] the random field distribution. However, some characteristics [of] these events such as the relative probability of an avalanche [of] the size of the avalanche may have a more general validity in th[e] thermodynamic limit.

3. Infinite-range ferromagnetic interactions

Models with short-range interactions are more realistic but al[so] more difficult to analyse theoretically. A qualitati[ve] understanding of the short-range interaction model may b[e] obtained through the analysis of a simplified version of [the] model in which a spin is allowed to interact equally with eve[ry] other spin. In this simplified version, the concept of a speci[al] lattice and its inter-connectivity disappears, and therefore th[e] analysis becomes much easier. If every spin interacts equall[y] with every other spin, we must scale down the pair interactio[n] proportionately to get an extensive energy for the system Le[t] $J = \tilde{J} / N$, where N is the number of spins in the system. In th[is] case,

$$h_i^{eff} = \tilde{J}\left[m(h) - \frac{s_i}{N}\right] + h_i + h ,$$

where $m(h)$ is the magnetization in applied field h:

$$m(h) = \frac{1}{N}\sum s_i .$$

The probability that $s_i = +1$ is the probability that $h_i^{eff} \geq 0$. In the thermodynamic limit, the second term in eq. (3) on the right hand side can be neglected. Thus the probability that $s_i = +1$ becomes a function of m only and can be written as $[1+m]/2$. For a Gaussian distribution of the random field with mean zero and variance σ^2, we get

$$m(h) = erf\left[\frac{\tilde{J}m(h)+h}{2\sigma^2}\right]. \tag{5}$$

The above equation has only one root if $\sigma \geq \sqrt{\frac{2}{\pi}}\tilde{J}$, but two roots otherwise. One root corresponds to magnetization in increasing field, and the other in decreasing field. The magnetization in increasing field h may be obtained by starting with $m = -1$ and iterating the above equation as usual in finding the root of an equation. For the other root, we may iterate the equation starting with $m = 1$. For the same value of applied field h, we find the magnetization in decreasing field is higher than the one in increasing field. This is to be expected from the phenomenology of hysteresis. We also find that in each half of the hysteresis loop, the magnetization makes a first order jump at a critical value of the applied field h_c that moves towards the origin with increasing σ. The size of the jump decreases with increasing σ, and vanishes continuously at $\sigma_c = \sqrt{\frac{2}{\pi}}\tilde{J}$. As $\sigma \to \sigma_c$, $h_c \to 0$. For $\sigma \geq \sigma_c$, *i.e.* $\sigma \geq 0.79\tilde{J}$ approximately, the two roots merge with each other. The common root has the symmetry $m(h) = -m(-h)$, and hence there is no hysteresis.

How do the above predictions compare with simulations of the short-range model on a cubic lattice ? At small values of σ, they are qualitatively similar. Simulations of the short-range RFIM on a simple cubic lattice produce hysteresis loops 'pleasantly familiar to the experimentalist' [1] There is a critical value of σ : $\sigma_c = 2.2J$ (approximately) on a simple cubic lattice. For $\sigma \leq \sigma_c$, each half of the hysteresis loop has a macroscopic jump at a critical applied field h_c. The jump decreases in size with increasing σ and vanishes continuously as σ approaches σ_c. At $\sigma = \sigma_c$ and $h = h_c$, the fluctuations in the size of avalanche become anomalously large, and scale invariant. In the mean field theory, the probability that an avalanche of size n occurs, scales algebraically as $n^{-1.5}$ In numerical simulations on a simple cubic lattice, it scales as $n^{-1.35\pm.2}$. At large values of σ, the behavior of the short-range model is qualitatively different from the mean field theory of the infinite–range model. The main difference is that, although the jump vanishes for $\sigma > \sigma_c$, the hysteresis does not. There is hysteresis for all values of σ although the area of the hysteresis loop decreases with increasing σ.

4. Ferromagnetic RFIM on Bethe lattice

Fluctuations in the short-range RFIM are qualitatively different from those in its infinite-range version. Consider the system in an increasing applied field h starting from a saturated state with all sites down. In the infinite-range version, all sites which are down at $h-\delta h$ have an equal *a priori* probability of flipping up at h. The reason is that all sites are directly connected to each other, and therefore have the same number of up and down neighbors. The only factor which decides which site flips up at h, is the random field at the site. This is not the case in the short-range RFIM. In the short-range model, each candidate that awaits turning up, does not have the same number of up and down neighbors. The probability that a site turns up, depends on the random field at the site as well as the number of its nearest neighbors which have already turned up. Thus the down sites in the short-range model do not turn up with the same probability. We must know how many nearest neighbors of a site are up before we can calculate the probability of its turning up.

Normally, if a nearest neighbor of a site is up, it causes a feedback influence to the site in question through closed loop paths. These feed-back effects are difficult to calculate. We therefore choose to work on a branching tree where such correlations are conveniently absent. Consider a Caley tree where each site has z nearest neighbors except sites at the boundary which have one nearest neighbor only. We are interested in the deep interior part of the tree far away from the boundaries. The deep interior of a Caley tree is also known as a Bethe lattice. On a Bethe lattice of coordination number z, each site is connected to z nearest neighbors by bonds. Each bond branches out into $z-1$ bonds at each end. On a Bethe lattice, two nearest neighbors of a site i have no way of interacting with each other except through the site i. Therefore, the probability $P^*(h)$ that a nearest neighbor of site i turns up at applied field h before site i turns up, is independent of the state of any other nearest neighbor of site i. In the thermodynamic limit, we may expect $P^*(h)$ to be independent of site i, and the same for each of its nearest neighbors.

The calculation of P^* proceeds as follows. We choose a site in the deep interior of a Caley tree, say site i, and focus on one of its nearest neighbors, say site j. The initial state has all spins pointing down. $P^*(h)$ is the probability that site j is up in increasing applied field h given that site i is down. Let site j be n steps from the nearest boundary where n is a large integer. Site j forms the vertex of a sub-tree of height n. Initially, all sites on the sub-tree are down. We relax the boundary sites first. The boundary sites can be relaxed independently of each other because they influence each other only through their common neighbor at height-1. The sites at height-1 are not relaxed in the first step. In the second step, we relax the sites at height-1 keeping the sites at height-2 down. Thus in the second step, sites at height-1 also can be relaxed independently of each other. This procedure can be repeated and yields the recursion relation

$$P^m(h) = \sum_{n=0}^{z-1}\binom{z-1}{n}\left[P^{m-1}(h)\right]^n\left[1-P^{m-1}(h)\right]^{z-1-n}p_n(h) \tag{6}$$

Here, P^m is the probability that a site at height-m is up given that its nearest neighbor at a higher height $m+1$ is down. Eq. (6) can be understood as follows. Each site has z nearest neighbors one of which (the one at a higher height) is kept down. This leaves $z-1$ neighbors at a lower height. Any $m(0 \leq m \leq z-1)$ of $z-1$ neighbors can be independently up with probability $P^{m-1}(h)$. The last factor $p_n(h)$ gives the probability that a site with n nearest neighbors up, and $z-n$ neighbors down has sufficient random field to flip it up at an applied field h. Eq. (6) iterates to the fixed point value $P^*(h)$ as $m \to \infty$. With the knowledge of $P^*(h)$, the probability that an arbitrary site i is up is easily calculated. We get,

$$p(h) = \sum_{n=0}^{z} \binom{z}{n} [P^*(h)]^n \left[1 - P^*(h)\right]^{z-n} p_n(h). \qquad (7)$$

The magnetization in increasing field h is given by $m(h) = 2p(h) - 1$. We can go all the way up to $h = \infty$ when all spins will be up and then reverse the field to get the upper half of the hysteresis loop. We need not do the calculation all over again to get the upper half of the loop. It is related to the lower half by a symmetry of the Ising model; $m_u(h) = -m_l(-h)$, where m_u and m_l denote respectively the magnetization on the upper and lower half of the major hysteresis loop. If the applied field is reversed before completing the lower half of the major hysteresis loop, we generate what is known as a minor hysteresis loop. First reversal of the field generates the upper half of the minor loop, and a second reversal generates the lower half. When the field on second reversal reaches the point where the first reversal was made, the lower half of the minor loop meets the starting point of the upper half. That is, the minor loop closes upon itself at the point it started. This property of the RFIM is known as return point memory.

The analytic calculation of minor loop is more difficult technically than that of the major loop. Consider the upper half of the minor loop. Suppose the applied field is reversed from h to $h'(h' \leq h)$. What we want to know is the probability that an arbitrary site i which was up at h turns down at h'. Naturally, it is necessary to know how many nearest neighbors of site i are up at h'. But this is not enough. We need to know separately how many of the up neighbors were up before site i turned up, and how many turned up after site i. When site i turns up, the field on each nearest neighbor increases by an amount $2J$. It does not affect the neighbors which are already up but some down neighbors may turn up as a result of the increased field. For each down neighbor that turns up after site i, the field on site i increases by an amount $2J$. Even if one down neighbor turns up, site i will not turn down if the applied field that causes it to turn up, is rolled back infinitesimally. It has to be rolled back sufficiently so that the neighbor which turned up after site i turns back down again. Site i can not turn down until all neighbors which turned up after it, have turned back down. This is the origin of hysteresis in the model, and also the difficulty in the calculation of the minor loop. Anyway, recognising the difficulty is the first step in solving it. The solution consists in calculati the probability $D^*(h')$ that a nearest neighbor of site i was down before site i turned up is down again at h'. $D^*(h$ is determined by the equation

$$\begin{aligned} D^*(h') = {} & \sum_{n=0}^{z-1} \binom{z-1}{n} \left[P_l^*(h)\right]^n \left[1 - P_l^*(h)\right]^{z-1-n} \left[1 - p_{n+1}(h)\right] \\ & + \sum_{n=0}^{z-1} \binom{z-1}{n} \left[P_l^*(h)\right]^n \left[D^*(h')\right]^{z-1-n} \\ & \times \left[p_{n+1}(h) - p_{n+1}(h')\right]. \end{aligned} \qquad ($$

Given a site i that is up at h, the first sum above, gives t conditional probability that a nearest neighbor of site i remai down at h after site i has turned up. The second sum takes ir account the situation that the nearest neighbor in question tu up at h after site i turns up but turns down at h'.

The fraction of up sites which turn down at h' is given

$$q'(h') = \sum_{n=0}^{z} \binom{z}{n} \left[P_l^*(h)\right]^n \left[D^*(h')\right]^{z-n} \left[p_n(h) - p_n(h')\right]. \quad ($$

The magnetization on the upper return loop is given by

$$m'(h') = 2\left[p(h) - q'(h')\right] - 1. \qquad (1$$

We reverse the field h' to h'' $(h'' > h')$ to trace the lov half of the return loop. The magnetization on the lower hal the return loop may be written as

$$m''(h'') = 2\left[p(h) - q'(h') + p''(h'')\right] - 1, \qquad (1$$

where $p''(h'')$ is the probability that an arbitrary site i whi turned up at h and turned down at h', turns up again at h''

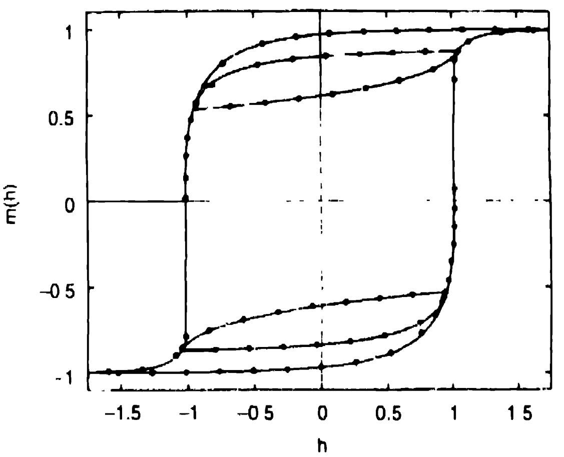
$$p''(h'') = \sum_{n=0}^{z} \binom{z}{n} \left[U^*(h'')\right]^n \left[D^*(h')\right]^{z-n} \left[p_n(h'') - p_n(h')\right]. \quad (1$$

Here, $U^*(h'')$ is the conditional probability that a near neighbor of a site i turns up before site i turns up on the lov return loop. It is determined by the equation

$$\begin{aligned} U^*(h'') = {} & P^*(h) \\ & - \sum_{n=0}^{z-1} \binom{z-1}{n} \left[P_l^*(h)\right]^n \left[D^*(h')\right]^{z-1-n} \left[p_n(h) - p_n(h')\right] \\ & + \sum_{n=0}^{z-1} \binom{z-1}{n} \left[U^*(h'')\right]^n \left[D^*(h')\right]^{z-1-n} \\ & \times \left[p_n(h'') - p_n(h')\right]. \end{aligned} \qquad (1$$

The rationale behind eq. (13) is similar to the one behind eq.) Given that a site i is down at h', the first two terms account r the probability that a nearest neighbor of site i is up at $' \geq h'$. Note that the neighbor in question must have been up h in order to be up at h', and if it is already up at h', then it ll remain up on the entire lower half of the return loop, *i.e.* at $' \geq h'$. The third term gives the probability that the ighboring site was down at h', but turned up on the lower urn loop before site i turned up.

It can be verified that the lower return loop meets the lower ajor loop at $h'' = h$ and merges with it for $h'' > h$, as may be pected on account of the return point memory. The exact pressions given above, have been checked against numerical nulations of the model in selected cases. A comparison is own in Figure 1 for the case of $z = 4$.



gure 1. Hysteresis at zero temperature in the RFIM (J = 1, σ = 1 70) a Bethe lattice with z = 4 The mean value of the random field is zero ere are macroscopic jumps in the major loop on Bethe lattices with > 1 The jumps decrease in size and move away from the origin with reasing σ As σ approaches a critical value σ_c, the jumps vanish at = 1 For z = 4, σ_c = 1 78 approximately Two minor loops are shown rting on the lower major loop at h = 0 95 and h = 1 05 respectively e minor loops meet the upper major loop when the applied field has en reversed by an amount $2J$ The theoretical result discussed in the t is shown by continuous lines Symbols show the data obtained from merical simulations of the model

Concluding remarks

he main message of the preceding analysis is that the zero mperature dynamics of the ferromagnetic RFIM can be solved actly on a Bethe lattice. The solution for coordination number ≥ 4 is qualitatively similar to the numerical simulation of the odel on a simple cubic lattice in the sense that there is a critical alue of disorder in both cases below which each half of the ysteresis loop has a macroscopic jump. It suggests that the ualitative features of the model are determined primarily by the coordination number of the lattice. The presence of closed loops on a d-dimensional lattice ($d > 1$), and their absence on a Bethe lattice, is of marginal importance. In order to test this idea, we have recently performed numerical simulations of RFIM on honeycomb and square lattices [8]. Both are two dimensional lattices but with different coordination numbers. The coordination number is equal to three for the honeycomb lattice, and four for the square lattice. We chose these lattices because the behavior on a Bethe lattice with coordination number $z = 3$ is qualitatively different from that on a lattice with $z = 4$. For $z = 3$, there is no critical value of a Gaussian disorder; the hysteresis loops have no macroscopic jump for any finite σ. On the other hand, there is a critical value of σ for $z = 4$, and hysteresis loops have jumps for $\sigma < \sigma_c$. We wanted to check if this difference persists between honeycomb and square lattice as well. Preliminary results show that it is indeed the case. We find no indication of a critical disorder on a honeycomb lattice. There is an indication of a critical disorder on a square lattice. The evidence for a critical disorder on a square lattice is not so strong as it is on a cubic lattice, but this is in accordance with the results of reference [2]. Thus, we may conclude that the RFIM on a Bethe lattice reproduces the qualitative behavior of RFIM on regular d-dimensional lattices.

The focus of this paper has been on ferromagnetic interactions. This is primarily because success in the analysis of dynamics of the anti-ferromagnetic RFIM has been rather limited so far. A solution of hysteresis in the anti-ferromagnetic RFIM has been obtained in one-dimension in a special case of the random field distribution [9]. In spite of effort over several years [10], we have not been able to extend this solution to a more general case. However, our efforts continue, and we hope that it would be possible in future to solve the anti-ferromagnetic RFIM as well on a Bethe lattice.

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